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2013 J. Phys. A: Math. Theor. 46 029501

(http://iopscience.iop.org/1751-8121/46/2/029501)

J. Phys. A: Math. Theor. **46** (2013) 029501 (2pp) doi:10.1088/1751-8113/46/2/029501

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2012 *J. Phys. A: Math. Theor.* **45** 015304

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Received 8 November 2012
Published 12 December 2012
Online at stacks.iop.org/JPhysA/46/029501

In section 5 of [1] we implicitly used the following lemma without proof.

Lemma. *The singular values of the* $M \times N$ *dimensional matrix* $F = P + g^n F_n$ *with* $M \leqslant N$ *have maximum leading order of* g^n, *where* $P = [p_1\vec{1} \cdots p_N\vec{1}]$ *and* $F_n = [\vec{E}_1 \cdots \vec{E}_N]$ *such that* $\sum_j p_j = 1$ *and* $\sum_j \vec{E}_j = \vec{0}$.

Proof. The M singular values of F are $\sigma_k = \sqrt{\lambda_k}$, where λ_k are the eigenvalues of the $M \times M$ dimensional matrix $G = FF^T$. The $N \times N$ dimensional matrix $H = F^T F$ also has the same M eigenvalues as G, as well as $(N - M)$ additional zero eigenvalues. Since $P^T F_n = 0$, the latter has the simple form $H = P^T P + g^{2n} F_n^T F_n$, where $(P^T P)_{ij} = M p_i p_j$ is $M||\vec{p}||^2$ times the projection operator onto the probability vector $\vec{p} = (p_1, \ldots, p_N)$ and $\left(F_n^T F_n\right)_{ij} = \vec{E}_i \cdot \vec{E}_j$. We will use H to determine the singular values of F.

Differentiating the eigenvalue relation $H(g^{2n})\vec{u}_k(g^{2n}) = \lambda_k(g^{2n})\vec{u}_k(g^{2n})$ with respect to g^{2n} produces the following deformation equation that describes how the eigenvalues of H continuously change with increasing g^{2n},

$$\dot{\lambda}_k(g^{2n}) = ||F_n \vec{u}_k(g^{2n})||^2. \tag{1}$$

Integrating this equation produces the following perturbative expansion of the eigenvalues for small g,

$$\lambda_k(g^{2n}) = \lambda_k(0) + g^{2n}||F_n \vec{u}_k(0)||^2 + O(g^{4n}). \tag{2}$$

Hence, to prove the lemma it is sufficient to show that $\lambda_k(0)$ and $||F_n\vec{u}_k(0)||$ cannot both vanish unless $\lambda_k(g^{2n}) = 0$ for all g.

Since $H(0) = P^T P$ is proportional to a projection operator, $\lambda_1(0) = M||\vec{p}||^2$ is its only nonzero eigenvalue with associated eigenvector $\vec{u}_1(0) = \vec{p}/||\vec{p}||$. Hence, $\sigma_1(g^{2n}) \approx \sqrt{M}||\vec{p}|| > 0$ to leading order. For $k \neq 1$, $\lambda_k(0) = 0$ and $\vec{u}_k(0)$ can be chosen arbitrarily to span the degenerate $(N - 1)$-dimensional subspace orthogonal to $\vec{u}_1(0)$. Suppose $||F_n\vec{u}_k(0)|| = 0$ for some $k \neq 1$, which implies $F_n\vec{u}_k(0) = 0$ since only the zero vector has zero norm. It follows that $H(g^{2n})\vec{u}_k(0) = P^T P\vec{u}_k(0) + g^{2n} F_n^T F_n \vec{u}_k(0) = 0$ since $\vec{u}_k(0)$ is orthogonal to $\vec{u}_1(0) \propto \vec{p}$. Therefore, $\vec{u}_k(0)$ is an eigenvector of $H(g^{2n})$ with eigenvalue 0 for any g. Since H is symmetric, its eigenvectors form an orthogonal set for any g, so we must have the identification $\vec{u}_k(g^{2n}) = \vec{u}_k(0)$. As a result, the associated eigenvalue vanishes for any g, $\lambda_k(g^{2n}) = \lambda_k(0) = 0$, which proves the lemma. □

This proof has also been included in a subsequent extended article [2].

Acknowledgments

We thank Dr Parrott for urging us to justify this lemma [3].

References

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